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Pointed Irreducible Bialgebras*

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INTRODUCTION

It is well-known [5, p. 274, Theorem 13.0.1] that a pointed irreducible cocommutative bialgebra B over a field of characteristic zero is isomorphic as a bialgebra to $U(P(B))$, the universal enveloping algebra of the Lie algebra $P(B)$ of primitives of B. One method of proof is first to show that B and $U(P(B))$ are each coalgebra-isomorphic to the cofree pointed irreducible cocommutative coalgebra on the vector space $P(B)$, and then to use these coalgebra isomorphisms to induce a bialgebra isomorphism.

In this paper we prove a dual version of the above. Let B be a pointed irreducible commutative bialgebra over a field of characteristic zero. Write I for its augmentation ideal, and $Q(B) = I/I^2$. Then B is isomorphic as a bialgebra to the appropriate universal object $U_1^c(Q(B))$. We first show that each is isomorphic as an algebra to $\mathrm{Sym}(Q(B))$, and then use these algebra isomorphisms to induce a bialgebra isomorphism.

The algebra isomorphism $B \xrightarrow{\sim} \mathrm{Sym}(Q(B))$ that we obtain is well-known, at least in the affine case. When B is affine, B represents a unipotent algebraic group scheme G. The Lie algebra L of G defines an affine scheme L_a, which is represented by $\mathrm{Sym}(L^*)$. There is a natural scheme isomorphism $\exp: L_a \xrightarrow{\sim} G$ [2, IV, Sect. 2, no. 4.1], and thus a natural isomorphism $B \xrightarrow{\sim} \mathrm{Sym}(L^*)$. The naturality allows the passage to the pro-affine case via direct limits, and the isomorphism so obtained is the same as the one we give here.

The corresponding bialgebra isomorphism exhibited in [5] for the cocommutative case is not natural. We sketch at the end of Section 2 how a natural isomorphism could be given in that case as well.

M. André has given in [1] a proof of the main result of this paper for the case in which B is graded, with B_0 the field. The methods of that paper are quite different from ours; the gradation facilitates inductive arguments.

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The basic terminology we use may be found in Sweedler's book [5]. We assume some familiarity with coalgebras and bialgebras, but include most of the definitions important for this paper.

I would like to take this opportunity to thank Professor Irving Kaplansky for his kind encouragement and many helpful conversations.

1. Preliminaries

Let V be a vector space over a field k. Recall that the tensor algebra $T(V)$ on V is defined as follows. As a (graded) vector space, we have $T(V)_0 = k$, $T(V)_1 = V$, $T(V)_n = \otimes^n V$, and $T(V) = \sum_{n=0}^{\infty} T(V)_n$. The restriction of the multiplication μ: $T(V) \otimes T(V) \to T(V)$ to $T(V)_n \otimes T(V)_m$ is given by $\mu((v_1 \otimes \cdots \otimes v_n) \otimes (v_{n+1} \otimes \cdots \otimes v_{n+m})) = v_1 \otimes \cdots \otimes v_{n+m}$. (When $n = 0$ or $m = 0$, μ is defined to be the canonical identification.) The unit of the algebra is the map $k \cong T(V)_0 \to T(V)$. The inclusion map i: $V = T(V)_1 \to T(V)$ has the following universal property: for every associative algebra A and every linear map f: $V \to A$, there exists a unique algebra map F: $T(V) \to A$ so that

$$\begin{tikzcd} T(V) \arrow[dr, "F"] & \\ V \arrow[u, "i"] \arrow[r, "f"'] & A \end{tikzcd}$$

commutes.

Note that $V \to T(V)$ is functorial. Given a linear transformation f: $V \to W$, we write $T_n(f)$ for the component of $T(f)$: $T(V) \to T(W)$ in degree n; thus, $T_n(f)(v_1 \otimes \cdots \otimes v_n) = f(v_1) \otimes \cdots \otimes f(v_n)$.

These properties allow us to endow $T(V)$ with a natural bialgebra structure. The linear map $V \to T(V) \otimes T(V)$ sending v to $v \otimes 1 + 1 \otimes v$ induces an algebra map Δ: $T(V) \to T(V) \otimes T(V)$. It is easy to use the universality to check that Δ is coassociative, with counit ϵ induced by the zero map $V \to k$. We remark that the uniqueness property tells us that if B is a bialgebra, then an algebra map F: $T(V) \to B$ will be a bialgebra map iff the algebra maps $(F \otimes F)\Delta$ and ΔF agree on V. That is, we require $\Delta F(v) = F(v) \otimes 1 + 1 \otimes F(v)$ for all $v \in V$; note that $\epsilon \otimes id$ applied to this relation gives us that $\epsilon F(v) = 0$, and thus that F preserves the counit.

The symmetric algebra $\mathrm{Sym}(V)$ on V is defined to be the quotient of $T(V)$ by the ideal generated by elements of the form $v_1 \otimes v_2 - v_2 \otimes v_1$, for $v_1, v_2 \in V$. $\mathrm{Sym}(V)$ has a universal property (as above) for associative *commutative* algebras, and the same formalism shows that it is a bialgebra.

Before giving the dual version of the above, we recall an important concept. A coalgebra C is said to be *pointed irreducible* if it has a unique minimal subcoalgebra C^0, and C^0 is one-dimensional. We will then denote by 1 the unique element of C^0 satisfying $\Delta 1 = 1 \otimes 1$ and $\epsilon(1) = 1$. For $n \geq 1$, set $C^n = \{c \in C: \Delta_n c \in 1 \otimes C \otimes C \otimes \cdots \otimes C + C \otimes 1 \otimes C \otimes \cdots \otimes C + C \otimes C \otimes 1 \otimes C \otimes \cdots \otimes C + \cdots + C \otimes \cdots \otimes C \otimes 1\}$. (Here $\Delta_1 = \Delta$, and $\Delta_n = (\Delta \otimes \mathrm{id})\Delta_{n-1}$.) Then $C^{n-1} \subset C^n$ and by [5, p. 185, Corollary 9.0.4] we have $C = \bigcup_{n \geqslant 0} C^n$. We remark that every coassociative coalgebra we encounter will be pointed irreducible, and turn to the tensor coalgebra.

The tensor coalgebra $T^c(V)$ on V (Sweedler's notation is: the shuffle algebra $\mathrm{Sh}(V)$) is defined as follows. The underlying vector space of $T^c(V)$ is the same as that of $T(V)$. The restriction of the comultiplication map Δ: $T^c(V) \to T^c(V) \otimes T^c(V)$ to $T^c(V)_n$ is given by: $\Delta(v_1 \otimes \cdots \otimes v_n) = 1 \otimes (v_1 \otimes \cdots \otimes v_n) + (v_1) \otimes (v_2 \otimes \cdots \otimes v_n) + (v_1 \otimes v_2) \otimes (v_3 \otimes \cdots \otimes v_n) + \cdots + (v_1 \otimes \cdots \otimes v_n) \otimes 1$. (For $n = 0$, we have $\Delta 1 = 1 \otimes 1$). The counit ϵ: $T^c(V) \to k$ is the projection onto degree zero. The projection map p: $T^c(V) \to T^c(V)_1 = V$ has the following universal property: for every pointed irreducible coassociative coalgebra C and every linear map $f: C \to V$ with $f(1) = 0$, there exists a unique coalgebra map $F: C \to T^c(V)$ so that

$$\begin{tikzcd} T^c(V) \arrow[d, "p"'] & \\ V & C \arrow[l, "f"] \arrow[ul, "F"'] \end{tikzcd}$$

commutes.

Indeed, we may exhibit $F = \epsilon + f + T_2(f)\Delta + \cdots + T_n(f)\,\Delta_{n-1} + \cdots = \sum_{n \geqslant 0} T_n(f)\,\Delta_{n-1}$ (where now $\Delta_{-1} = \epsilon$, $\Delta_0 = id$). Since $T_n(f)\,\Delta_{n-1}$ vanishes on C^{n-1}, the sum is finite when evaluated at any element of C.

Note that $V \to T^c(V)$ is functorial.

The linear map $f: T^c(V) \otimes T^c(V) \to V$ sending $a \otimes b$ to $p(a)\,\epsilon(b) + \epsilon(a)p(b)$ defines a coalgebra map μ: $T^c(V) \otimes T^c(V) \to T^c(V)$. It is easy to use the universality to show that μ is associative, and that the element 1 of $T^c(V)_0$ is a unit for μ. Thus $T^c(V)$ is a bialgebra. Using the formula above, we have that $p \circ \mu = p \circ (\epsilon + f + T_2(f)\Delta + \cdots) = f$. Thus, for any $a, b \in T^c(V)$, we have $p(ab) = f(a \otimes b) = p(a)\,\epsilon(b) + \epsilon(a)\,p(b)$—i.e., p is an ϵ-derivation. Now for any coalgebra C, the twist map τ: $C \otimes C \to C \otimes C$ sending $c \otimes d$ to $d \otimes c$ is a coalgebra map. For $a, b \in T^c(V)$ we have $p\mu(a \otimes b) = p\mu\tau(a \otimes b)$. By uniqueness, $\mu = \mu\tau$—i.e., the multiplication on $T^c(V)$ is commutative.

A further consequence of uniqueness is that if B is a bialgebra, then a coalgebra map F: $B \to T^c(V)$ is a bialgebra map iff the map $B \otimes B \to^{F \otimes F} T^c(V) \otimes T^c(V) \to^{\mu} T^c(V) \to^{p} V$ agrees with $B \otimes B \to^{\mu} B \to^{F} T^c(V) \to^{p} V$. This reduces to the requirement that the map $f = p \circ F$: $B \to V$ be an ϵ-derivation; since F is a coalgebra map, $F(1) = 1$ is automatic.

We write $\mathrm{Sym}^c(V)$ for the subcoalgebra of $T^c(V)$ spanned by the symmetric tensors. $\mathrm{Sym}^c(V)$ is a subbialgebra of $T^c(V)$, and has a universal property for pointed irreducible *cocommutative* coalgebras.

We consider the case $\dim V = 1$. Then $T(V) = \mathrm{Sym}(V)$, and $T^c(V) = \mathrm{Sym}^c(V)$. Write $V = kv$, $v_0 = 1$, $v_n = \otimes^n v$. Then $T(V)$ has the structure: $v_1{}^n = v_n$, $\Delta v_n = \sum_{r+s=n} \binom{n}{r} v_r \otimes v_s$. The structure of $T^c(V)$ is given by: $v_1{}^n = n! v_n$, $\Delta v_n = \sum_{r+s=n} v_r \otimes v_s$. When characteristic $k = 0$ (and $\dim V = 1$), the natural bialgebra map $T(V) \to T^c(V)$ is an isomorphism.

Let C be a coalgebra, A an algebra. Then $\mathrm{Hom}(C, A)$-the vector space of k-linear maps from C to A— has a "convolution" algebra structure given by $f * g = \mu(f \otimes g)\Delta$. The identity is the map $e(c) = \epsilon(c)1_A$. A coalgebra map $C \to C'$, or an algebra map $A \to A'$, will induce an algebra map on the appropriate convolution algebra.

Now take C to be pointed irreducible. Then $\mathscr{M} = \{f \in \mathrm{Hom}(C, A) : f(1) = 0\}$ is an ideal. If $f \in \mathscr{M}^n$, then we see that f will vanish on C^{n-1}. Thus $\mathrm{Hom}(C, A)$ is complete in its $\mathscr{M}$-adic topology. Note that, for C, C' pointed irreducible, the induced algebra maps mentioned above will be continuous.

We now assume that characteristic $k = 0$.

If $F \in \mathrm{Hom}(C, A)$ and $F(1) = 1$, then $e - F \in \mathscr{M}$. Thus we can define $\log F = \log(e - (e - F)) = -\sum_{n=1}^{\infty} (e - F)^n/n$. For every pointed irreducible bialgebra B over a field of characteristic zero, we define $D_B = \log id_B$. Note that $D_B(1) = 0$, and $\epsilon \circ D_B = 0$.

LEMMA 1. *B, B' pointed irreducible bialgebras, characteristic $k = 0$, F: $B \to B'$ linear.*

(i) *If F is an algebra map and $F \circ e = e$, then* $\log F = F \circ D_B$

(ii) *If F is a coalgebra map, then* $\log F = D_{B'} \circ F$.

Proof.

(i)
$$
\begin{aligned}
F \circ D_B &= F\left(-\sum_1^{\infty} \frac{1}{n}(e - id_B)^n\right) \\
&= -\sum_1^{\infty} \frac{1}{n} F(e - id_B)^n \qquad \text{by continuity} \\
&= -\sum_1^{\infty} \frac{1}{n} (F(e - id_B))^n \\
&= -\sum_1^{\infty} \frac{1}{n} (e - F)^n = \log F.
\end{aligned}
$$

The proof of (ii) is dual. ∎

We now have that if F: $B \to B'$ is a bialgebra map, then $F \circ D_B = \log F = D_{B'} \circ F$; that is, D is natural. We will often drop the subscript B.

If $G \in \mathrm{Hom}(C, A)$ satisfies $G(1) = 0$, then we may define $\exp G = \sum_{n=0}^{\infty} G^n/n!$.

LEMMA 2. $F \in \operatorname{Hom}(C, A), F(1) = 1$.

$$\textit{Then:} \quad \exp(\log F) = F.$$

Proof. Let t be an indeterminate. We know that $\exp(\log(1 - t)) = 1 - t$ in $k[[t]]$. Since $e - F \in \mathscr{M}$, there is a continuous algebra map $\varphi: k[[t]] \to \operatorname{Hom}(C, A)$ sending t to $e - F$. Applying φ to the above identity, we obtain $\exp(\log F) = F$. ∎

Thus we have, in particular, that $\exp(D_B) = id_B$. We note for future use that the same technique will show that $\log(F^n) = n \log F$, n a non-negative integer.

PROPOSITION 3. *B pointed irreducible bialgebra, characteristic $k = 0$.*

(i) *The natural algebra map*

$$\begin{tikzcd} T(\operatorname{im} D) \arrow[dr, dashed, "\Psi"] & \\ \operatorname{im} D \arrow[u] \arrow[r] & B \end{tikzcd}$$

is surjective.

(ii) *The natural coalgebra map*

$$\begin{tikzcd} T^c(\operatorname{coim} D) \arrow[d] & \\ \operatorname{coim} D & B \arrow[l] \arrow[ul, dashed, "\Phi"'] \end{tikzcd}$$

is injective.

Proof.

(i) For $b \in B$ we have $D^n b = \sum (Db_1) \cdots (Db_n) \in \operatorname{im} \Psi$. Thus $b = id_B b = (\exp D)b = \sum D^n b/n! \in \operatorname{im} \Psi$, and Ψ is surjective.

(ii) Write $q: B \to \operatorname{coim} D$, $i: \operatorname{coim} D \simeq \operatorname{im} D \to B$. Then $D = i \circ q$. Recall $\Phi = \sum_{n=0}^{\infty} T_n(q) \Delta_{n-1}$. Then

$$\begin{aligned} \Phi b = 0 &\Rightarrow T_n(q) \Delta_{n-1} b = 0 \qquad \text{all } n \\ &\Rightarrow T_n(i) T_n(q) \Delta_{n-1} b = 0 \qquad \text{all } n \\ &\Rightarrow D^n b = 0 \qquad \text{all } n \\ &\Rightarrow b = (\exp D) b = 0. \end{aligned}$$

Thus Φ is injective. ∎

2. The algebra isomorphism

We will now specialize to the case B commutative.

LEMMA 4. *V a vector space, $f \in V \otimes k[t] = V[t]$. If f has infinitely many zeros in k, then $f = 0$.*

Proof. Say $f = v_0 + v_1 t + \cdots + v_n t^n$.

Define $F\colon k^{n+1} \to V$ by $F((\lambda_0, \ldots, \lambda_n)) = \sum \lambda_i v_i$. Clearly $\ker F$ is a subspace of k^{n+1}. If α is a zero of F, then $(1, \alpha, \alpha^2, \ldots, \alpha^n) \in \ker F$. By Vandermonde, $n+1$ distinct zeros of F will give us $n+1$ linearly independent elements of $\ker F$. Thus $\ker F = k^{n+1}$, and $v_i = F((0, \ldots, 1, \ldots, 0)) = 0$. ∎

PROPOSITION 5. *D is an ϵ-derivation.*

Proof. Let t be an indeterminate, $a, b \in B$. If we evaluate the expression $\exp(tD)(ab) - \exp(tD)(a)\exp(tD)(b)$ for t a non-negative integer, we obtain $id^t(ab) - id^t(a)\, id^t(b)$, since we then have $\exp(tD) = \exp(t \log id_B) = \exp \log id_B{}^t = id_B{}^t$. Since B is commutative, the convolution product of algebra maps $B \to B$ is again an algebra map; thus, $id^t(ab) = id^t(a)\, id^t(b)$. It follows from Lemma 4 that every coefficient in the expression

$$\sum_n \left(D^n(ab) - \sum_{r+s=n} \binom{n}{r} D^r(a)\, D^s(b) \right) \frac{t^n}{n!}$$

must vanish. In particular, we have for $n = 1$ that $D(ab) = \epsilon(a)D(b) + D(a)\epsilon(b)$. ∎

Write $I = \ker \epsilon$.

PROPOSITION 6.

(i) $\ker D = k1 + I^2$

(ii) *For all $b \in B$, we have $b - \epsilon(b)1 - D(b) \in I^2$*

(iii) $D \circ D = D$.

Proof. By Proposition 5, $k1 + I^2 \subset \ker D$. Now consider the relation

$$b = (\exp D)b = \epsilon(b)1 + D(b) + \tfrac{1}{2}D^2(b) + \tfrac{1}{6}D^3(b) + \cdots.$$

Since $D^n B \subset I^n$, we see that (ii) holds. This gives us that $\ker D \subset k1 + I^2$, and we have (i). Part (iii) follows directly from (i) and (ii). ∎

We now see that for V one-dimensional, $D\colon T^c(V) \to T^c(V)$ is simply the projection onto degree one.

Write $Q(B) = I/I^2$. We have a surjection $q = q_B\colon B \to Q(B)$, sending b to the image of $b - \epsilon(b)1$. By Proposition 6 (part (i)), we may identify this map with $B \to \operatorname{coim} D$. We will write

$$i_B\colon Q(B) \to B \qquad \text{for the map} \qquad \operatorname{coim} D \simeq \operatorname{im} D \to B.$$

Thus $i_B \circ q_B = D_B$. Note that by parts (iii) and (i) above, we have $b - D(b) \in \ker D = \ker q$, and thus $q \circ D = q$.

LEMMA 7: *B a bialgebra, A a commutative algebra, $f\colon B \to A$ an ϵ-derivation. Then $f^n \circ \mu = \mu \circ (\sum_{r+s=n} \binom{n}{r} f^r \otimes f^s)$. If B is pointed irreducible and characteristic $k = 0$, then $\exp f$ is an algebra map.*

Proof. For $a, b \in D$ we have

$$\begin{aligned} f^n(ab) &= \sum f(a_1b_2) f(a_2b_2) \cdots f(a_nb_n) \\ &= \sum (\epsilon(a_1) f(b_1) + f(a_1)\,\epsilon(b_1))(\epsilon(a_2) f(b_2) + f(a_2)\,\epsilon(b_2)) \\ &\qquad\qquad\qquad \cdots (\epsilon(a_n) f(b_n) + f(a_n)\,\epsilon(b_n)) \\ &= \sum_{r+s=n} \sum_{\substack{i_1<\cdots<i_r \\ i_{r+1}<\cdots<i_{r+s}}} f(a_{i_1})\epsilon(b_{i_1}) \cdots f(a_{i_r})\epsilon(b_{i_r})\epsilon(a_{i_{r+1}})f(b_{i_{r+1}}) \cdots \epsilon(a_{i_{r+s}})f(b_{i_{r+s}}) \\ &= \sum_{r+s=n} \sum_{\substack{i_1<\cdots<i_r \\ i_{r+1}<\cdots<i_{r+s}}} f^r(a) f^s(b) \\ &= \sum_{r+s=n} \binom{n}{r} f^r(a) f^s(b). \end{aligned}$$

The second statement follows immediately from this. ∎

THEOREM 8. *B a pointed irreducible commutative bialgebra, characteristic $k = 0$. Then: the natural algebra map*

$$\begin{tikzcd} \mathrm{Sym}(Q(B)) \arrow[dr, "\psi"] & \\ Q(B) \arrow[u, "i_s"] \arrow[r, "i_B"] & B \end{tikzcd}$$

is an algebra isomorphism.

Proof. Note that $i_s \circ q$ is an ϵ-derivation; thus, the map $\theta = \exp(i_s \circ q)\colon B \to \mathrm{Sym}(Q)$ is an algebra map. We will show that θ is an inverse for ψ. Since ψ is an algebra map, we have

$$\psi((i_s \circ q)^n) = (\psi \circ i_s \circ q)^n = D^n, \qquad \text{all } n.$$

Thus $\psi \circ \theta = \psi \circ \exp(i_s \circ q) = \exp D = id_B$.

In order to show $\theta \circ \psi = id_{\mathrm{Sym}(Q)}$, it suffices to show $\theta \circ i_B = i_s$; for then we have $\theta \circ \psi \circ i_s = \theta \circ i_B = i_s$, and we are done. Thus, we want $\theta(Db) = i_s \circ q(b)$ for all $b \in B$. We need $(i_s \circ q)^n(Db) = 0$ for $n \geqslant 2$.

Now $(i_s \circ q)^n(Db)$ is the image under the natural map h: $T(Q)_n \to \mathrm{Sym}(Q)_n$ of the element $T_n(q)\, \Delta_{n-1}Db$. Ker h consists of those elements z for which $T_n(f)z = 0$, all $f: Q \to V$, $V = kv$ one-dimensional.

Since $f \circ q$ is an ϵ-derivation, the map $F = \sum_{n\geqslant 0} T_n(f \circ q)\, \Delta_{n-1}$: $B \to T^c(V)$ is a bialgebra map. Thus $F(Db) = DF(b) \in T^c(V)_1$. Thus for $n \geqslant 2$, we have $0 = T_n(f \circ q)\, \Delta_{n-1}Db = T_n(f)\, T_n(q)\, \Delta_{n-1}Db$, as required. ▌

Remark. Suppose that B is a pointed irreducible cocommutative bialgebra, characteristic $k = 0$. Then the dual of the argument of Proposition 5 shows that D is an ϵ-coderivation—i.e. $D = (\epsilon \otimes D + D \otimes \epsilon)\Delta$. This shows that $D(B) \subset P(B)$, the primitive elements of B. On the other hand, a trivial calculation using the series definition of D shows that D acts as the identity on $P(B)$; thus D is a projection onto $P = P(B)$. D defines a natural coalgebra map φ from B to $\mathrm{Sym}^c(P)$. The map from $\mathrm{Sym}^c(P)$ to B obtained by exponentiating the ϵ-coderivation $\mathrm{Sym}^c(P) \to P \to B$ is a coalgebra map as in Lemma 7. It is the inverse to φ by the argument dual to that given in Theorem 8.

3. The Lie Structure

Let C be a coalgebra. The Lie coalgebra associated to C is the vector space C together with the map $[\ ]$: $C \to C \otimes C$ defined by: $[\ ] = \Delta - \tau\Delta$, where τ: $C \otimes C \to C \otimes C$ is the twist map. We write $[c] = \sum c_1 \otimes c_2 - \sum c_2 \otimes c_1$. More generally, a Lie coalgebra is a vector space V equipped with a linear transformation $[\ ]$: $V \to V \otimes V$ for which there exists a coalgebra C and a surjection q: $C \to V$ for which

$$\begin{array}{ccc} C & \xrightarrow{[\ ]} & C \otimes C \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle q\otimes q} \\ V & \xrightarrow{[\ ]} & V \otimes V \end{array}$$

commutes.

Then q becomes a Lie coalgebra map.

(Note that the fundamental theorem on coalgebras [5, p. 46, Theorem 2.2.1] implies that every Lie coalgebra is the union of its finite-dimensional subcoalgebras. It is possible to extend the notion of a Lie coalgebra to include objects which to not have this finiteness property.)

Let B be a bialgebra, I the augmentation ideal, q: $B \to Q(B) = I/I^2$ the map $q(b) = \overline{b - \epsilon(b)1}$. It is easy to check that the formula $[q(b)] = (q \otimes q)[b]$ defines on $Q(B)$ the structure of a Lie coalgebra.

Motivated by the notion of a universal enveloping algebra of a Lie algebra, we make the following definition.

Let Q be a Lie coalgebra. A universal pointed irreducible coalgebra of Q is a Lie coalgebra map $j\colon U_1{}^c(Q) \to Q$, where $U_1{}^c(Q)$ is a pointed irreducible coalgebra, such that: for every pointed irreducible coalgebra C and every Lie map $C \to Q$ sending 1 to 0, there exists a unique coalgebra map $C \to U_1{}^c(Q)$ so that

$$\begin{tikzcd} U_1{}^c(Q) \arrow[d, "j"] & \\ Q & C \arrow[l] \arrow[ul] \end{tikzcd}$$

commutes.

It is easy to see that if $U_1{}^c(Q)$ exists, then it is unique up to unique isomorphism. (By considering $Q \leftarrow k1$, we obtain $j(1) = 0$). We can construct $U_1{}^c(Q)$ as the sum of all subcoalgebras of the tensor coalgebra $T^c(Q)$ on which the restriction j of $p\colon T^c(Q) \to Q$ is a Lie coalgebra map. However, it is not clear that $U_1{}^c(Q)$ must be any larger than the field k.

We note that $\operatorname{coim} j$ is a Lie coalgebra, and that there is a natural isomorphism $U_1{}^c(Q) \simeq U_1{}^c(\operatorname{coim} j)$.

We can check directly that if $q\colon C \to V$ and $q'\colon C' \to V'$ are Lie maps from coassociative coalgebras to Lie coalgebras, then the map $C \otimes C \to^{q \otimes \epsilon + \epsilon \otimes q'} V \otimes k + k \otimes V' \simeq V \oplus V'$ is a Lie map. Also, if V is a Lie coalgebra, then the vector-space addition map $a\colon V \oplus V \to V$ is a Lie map. Thus we have a Lie map $U_1{}^c(Q) \otimes U_1{}^c(Q) \to^{j \otimes \epsilon + \epsilon \otimes j} Q \otimes k + k \otimes Q \simeq Q \oplus Q \to^{a} Q$. This defines a multiplication $\mu\colon U_1{}^c(Q) \otimes U_1{}^c(Q) \to U_1{}^c(Q)$. Notice that $j \circ \mu$ is simply the restriction to $U_1{}^c(Q) \otimes U_1{}^c(Q)$ of the map $T^c(Q) \otimes T^c(Q) \to Q$ which induces the algebra structure on $T^c(Q)$. Thus, μ is simply the restriction of the multiplication map of $T^c(Q)$ to $U_1{}^c(Q)$, and, in particular, $U_1{}^c(Q)$ is a subbialgebra of $T^c(Q)$. Since p is an ϵ-derivation, j is an ϵ-derivation as well.

LEMMA 9. *Q a Lie coalgebra, C a pointed irreducible coalgebra, characteristic $k = 0$. Let $f\colon C \to Q$ be a Lie coalgebra map with $f(1) = 0$. Then the maps*

$$C \xrightarrow{F} T^c(Q)$$

and

$$C \xrightarrow{F} T^c(Q) \xrightarrow{\exp(i_s \circ p)} \operatorname{Sym}(Q)$$

have the same kernel.

Proof. Since F is a coalgebra map, we have $\exp(i_s \circ p) \circ F = \exp(i_s \circ p \circ F) = \exp(i_s \circ f)$. Thus $(\exp(i_s \circ p) \circ F)(c) = \epsilon(c)1 + i_s f(c) + \frac{1}{2} \sum i_s f(c_1)\, i_s f(c_2) + \cdots$. We must show that if this is zero, then we already have $0 = F(c) = \epsilon(c)1 +$

$f(c) + \sum f(c_1) \otimes \sum f(c_2) + \cdots$. It is apparent that the first two terms will vanish. We proceed inductively. Suppose $\sum f(c_1) \otimes \cdots \otimes f(c_r) = 0$. Then

$$\begin{aligned}
0 &= \sum [f(c_1)] \otimes f(c_2) \otimes \cdots \otimes f(c_r) \\
&= \sum (f \otimes f)[c_1] \otimes f(c_2) \otimes \cdots \otimes f(c_r) \\
&= \sum (f(c_1) \otimes f(c_2) - f(c_2) \otimes f(c_1)) \otimes f(c_3) \otimes \cdots \otimes f(c_{r+1}).
\end{aligned}$$

Since we can repeat this at any position, $\sum f(c_1) \otimes f(c_2) \otimes \cdots \otimes f(c_{r+1})$ is symmetric. But, by assumption, the image of this element in $\mathrm{Sym}(Q)$ is zero. Thus it is itself zero. ∎

PROPOSITION 10. *If j: $U_1{}^c(Q) \to Q$ is surjective, then*

$$\exp(i_s \circ j)\colon U_1(Q) \to \mathrm{Sym}(Q)$$

is an algebra isomorphism.

Proof. By Lemma 7, $\exp(i_s \circ j)$ is an algebra map. Let i: $U_1{}^c(Q) \to T^c(Q)$ be the inclusion. We have $\exp(i_s \circ j) = \exp(i_s \circ p \circ i) = \exp(i_s \circ p) \circ i$. Thus, by Lemma 9 we have that $\exp(i_s \circ j)$ is injective. By Lemma 1, we have $\exp(i_s \circ j) \circ D = \log \exp(i_s \circ j) = i_s \circ j$. We see that the image of $\exp(i_s \circ j)$ contains Q, and thus that $\exp(i_s \circ j)$ is surjective. ∎

Remark. Since j is an ϵ-derivation, we have $\ker j \supset k1 + I^2$. When j is surjective, the algebra map $\exp(i_s \circ j)$ sends I to the ideal J of $\mathrm{Sym}(Q)$ generated by Q. If $x \in I \backslash I^2$, then we must have $\exp(i_s \circ j)(x) \in J \backslash J^2$; thus, $j(x) \neq 0$. This gives us that $\ker j = k1 + I^2$. Since j is Lie coalgebra map, we conclude that $Q(U_1{}^c(Q)) \simeq Q$ when j is surjective. More generally, $Q(U_1{}^c(Q)) \simeq \operatorname{coim} j$.

THEOREM 11. *B a pointed irreducible commutative bialgebra, characteristic $k = 0$. Then the natural coalgebra map $B \to U_1{}^c(Q(B))$ is a bialgebra isomorphism.*

Proof. Write $Q = Q(B)$. We have

$$\begin{tikzcd}
\mathrm{Sym}(Q) & T^c(Q) \arrow[l, "\exp(i_s \circ p)"'] & B \arrow[l, "\Phi"'] \arrow[dl, "h"] \\
& U_1{}^c(Q) \arrow[ul, "\exp(i_s \circ j)"] \arrow[u, hook, "i"'] &
\end{tikzcd}$$

Since $p \circ \Phi = q\colon B \to Q$ is an ϵ-derivation, Φ is a bialgebra map. Thus, h is a bialgebra map. Now $\exp(i_s \circ p) \circ \Phi = \exp(i_s \circ p \circ \Phi) = \exp(i_s \circ q)$ is our algebra isomorphism θ of Theorem 8. Since $\exp(i_s \circ j)$ is an algebra isomorphism by Proposition 10, h is an isomorphism. ∎

Remark. If $f: C \to Q(B)$ is a Lie map, where C is a pointed irreducible coalgebra and $f(1) = 0$, then by Theorem 11 there is a unique coalgebra map $F: C \to B$ so that

$$\begin{tikzcd} B \arrow[d, "q"'] & \\ Q & C \arrow[l, "f"] \arrow[ul, "F"'] \end{tikzcd}$$

commutes.

Applying our algebra isomorphism θ of Theorem 8, we calculate that $\theta \circ F = \exp(i_s \circ q) \circ F = \exp(i_s \circ q \circ F) = \exp(i_s \circ f) = \exp(\theta \circ i_B \circ f) = \theta \circ (\exp(i_B \circ f))$. Thus $F = \exp(i_B \circ f)$.

THEOREM 12. *B, B' pointed irreducible commutative bialgebras, characteristic $k = 0$. Then: for every Lie map $f: Q(B) \to Q(B')$ there exits a unique bialgebra map $F: B \to B'$ so that*

$$\begin{tikzcd} B \arrow[r, "F"] \arrow[d, "q_B"] & B' \arrow[d, "q_{B'}"] \\ Q(B) \arrow[r, "f"] & Q(B') \end{tikzcd}$$

commutes.

Moreover, in this case

$$\begin{tikzcd} B \arrow[r, "F"] & B' \\ Q(B) \arrow[r, "f"] \arrow[u, "i_B"] & Q(B') \arrow[u, "i_{B'}"] \end{tikzcd}$$

commutes.

Proof. By the above, remark, we get a unique coalgebra map $F = \exp(i_{B'} \circ f \circ q_B)$; F is an algebra map by Lemma 7. Moreover, we have $F \circ i_B \circ q_B = F \circ D_B = \log F = \log \exp(i_{B'} \circ f \circ q_B) = i_{B'} \circ f \circ q_B$. Thus $F \circ i_B = i_{B'} \circ f$. ∎

4. EXISTENCE

A Lie coalgebra Q occurs as $Q = Q(B)$ for some pointed irreducible commutative bialgebra B over a field of characteristic zero iff the map $j: U_1^c(Q) \to Q$ is surjective. In this section, we establish when this is the case.

Let C be a pointed irreducible coalgebra, $x \in C^n$. It can readily be shown [5, p. 201, Proposition 10.0.2] that $\Delta x - 1 \otimes x \otimes - x \otimes 1 \in C^{n-1} \otimes C^{n-1}$. Thus $[x] \in C^{n-1} \otimes C^{n-1}$. It follows that any Lie coalgebra which is a Lie image of C will have a nilpotence property.

Write $[\ ]_n = (id \otimes [\ ]_{n-1})[\ ]$. We say that a Lie coalgebra Q is nilpotent if there exists n with $[x]_n = 0$ all $x \in Q$. Q is locally nilpotent if every finite-dimensional subcoalgebra is nilpotent.

We observe that if j: $U_1{}^c(Q) \to Q$ is onto, then Q is locally nilpotent.

LEMMA 13. *L a Lie algebra. Write $L_1 = L$, $L_{r+1} = [L, L_r]$ for $r \geqslant 1$. Let $U(L)$ be the universal enveloping algebra of L, and I the ideal of $U(L)$ generated by L. Then: if $L_n = 0$, we have $L \cap I^n = 0$.*

Proof. We have $0 = L_n \subset L_{n-1} \subset L_{n-2} \subset \cdots \subset L_1 = L$. Take a basis $\{x_{ij}\}$ for L so that, for each i, $\{x_{ij}\}$ is a basis of L_i modulo L_{i+1}. Recall [3, p. 13, Theorem 7] that $[L_r, L_s] \subset L_{r+s}$. Thus, when we write $[x_{ij}, x_{i'j'}] = \sum \lambda_{uv} x_{uv}$, we find that $u \geqslant i + i'$ whever $\lambda_{uv} \neq 0$. It follows easily that whenever a monomial of degree $\geqslant n$ is put into its Poincare-Birkhoff-Witt standard form, no linear terms will appear. Such monomials span I^n. ∎

THEOREM 14. *Q a Lie coalgebra, characteristic $k = 0$. Then: Q occurs as $Q = Q(B)$ for some pointed irreducible commutative bialgebra B iff Q is locally nilpotent.*

Proof. We must show that if Q is locally nilpotent, then j: $U_1{}^c(Q) \to Q$ is onto. It suffices to show that if Q is finite-dimensional, then there exists a finite-dimensional pointed irreducible coalgebra C and a Lie surjection $C \to Q$ sending 1 to 0. Dually, we must show that if Q^* is a finite-dimensional nilpotent Lie algebra, then there exists a finite-dimensional algebra C^* with radical of codimension one and a Lie injection $Q^* \to C^*$ sending Q^* into the radical. We may take $C^* = U(Q^*)/I^n$, where $U(Q^*)$ is the universal enveloping algebra of Q^*, I its augmentation ideal, and n is large enough. ∎

5. CALCULATION OF THE COPRODUCT

We have shown (Theorem 14; Proposition 10) that if Q is a locally nilpotent Lie coalgebra over a field of characteristic zero, then the algebra Sym(Q) has a pointed irreducible bialgebra structure, with $Q(\mathrm{Sym}(Q)) \simeq Q$ as a Lie coalgebra. In this section we write down a formula for the coproduct of Sym(Q) in terms of the Lie structure of Q.

For any pointed irreducible bialgebra B over a field of characteristic zero, we may define maps X, Y: $B \to B \otimes B$ by: $X(b) = Db \otimes 1$, $Y(b) = 1 \otimes Db$. We find that $X^r Y^s = (D^r \otimes D^s)\Delta$, and thus $(\exp X)(\exp Y) = \Delta$. By Lemma 1, $\Delta D = \log \Delta = \log(\exp X \exp Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y] + [Y, [Y, X]]) + \cdots$—the Campbell-Hausdorff formula [4, LA 4.15].

The formula involves terms such as $(\mathrm{ad}\, X)^{e_1}(\mathrm{ad}\, Y)^{f_1} \cdots (\mathrm{ad}\, X)^{e_r} Y$ (of degree $n = e_1 + f_1 + \cdots + e_r + 1$), and similar terms with X and Y interchanged. We show how to evaluate a term at any $b \in B$. First, write down the formal product $[X_1[X_2[\cdots[X_{n-1}X_n]]\cdots]$. Let a_σ be the coefficient of the term $X_{\sigma(1)}\cdots X_{\sigma(n)}$, σ an element of the symmetric group S_n on $\{1,\ldots,n\}$. Suppose X occurs in $X^{e_1}Y^{f_1}\cdots X^{e_r}Y$ in positions $u_1 < u_2 < \cdots < u_s$. Represent these positions (upon re-ordering) as $\sigma^{-1}(j_1),\ldots,\sigma^{-1}(j_s)$, where $j_1 < \cdots < j_s$. Write the remaining positions $v_1 < \cdots < v_{n-s}$ as $\sigma^{-1}(i_1),\ldots,\sigma^{-1}(i_{n-s})$, where $i_1 < \cdots < i_{n-s}$. Then we find that $(\mathrm{ad}\,X)^{e_1}(\mathrm{ad}\,Y)^{f_1}\cdots(\mathrm{ad}\,X)^{e_r}Y(b) = \sum_{\sigma\in S_n} a_\sigma \sum Db_{j_1}\cdots Db_{j_s}\otimes Db_{i_1}\cdots Db_{i_{n-s}}$.

Recall that the map $q\colon B\to Q(B) = I/I^2$ is a Lie coalgebra map. Thus $[q(b)]_{n-1} = \sum_{\sigma\in S_n} a_\sigma q(b_{\sigma(1)})\otimes\cdots\otimes q(b_{\sigma(n)}) \in \otimes^n Q(B)$. Now assume in addition that B is commutative. Then we have a map $i_B\colon Q(B)\to B$ with $i_B\circ q = D$. Thus

$$T_n(i_B)[q(b)]_{n-1} = \sum_{\sigma\in S_n} a_\sigma Db_{\sigma(1)}\otimes\cdots\otimes Db_{\sigma(n)}.$$

Let $m_{u,v}\colon \otimes^n B\to B\otimes B$ be the multiplication map which sends the entries in positions $u_1,\ldots,u_s$ to the left factor, and the entries in positions $v_1,\ldots,v_{n-s}$ to the right. Then

$$m_{u,v}T_n(i_B)[q(b)]_{n-1} = \sum_{\sigma\in S_n} a_\sigma Db_{\sigma(u_1)}\cdots Db_{\sigma(u_s)}\otimes Db_{\sigma(v_1)}\cdots Db_{\sigma(v_{n-s})}.$$

Since (for each σ) $j_1,\ldots,j_s$ is a re-ordering of $\sigma(u_1)\cdots\sigma(u_s)$, and $i_1,\ldots,i_{n-s}$ is a re-ordering of $v_1,\ldots,v_{n-s}$, we have that $(\mathrm{ad}\,X)^{e_1}(ad\,Y)^{f_1}\cdots(\mathrm{ad}\,X)^{e_r}Y(b) = m_{u,v}T_n(i_B)[q(b)]_{n-1}$.

We now consider the bialgebra structure we have defined on $\mathrm{Sym}(Q)$. Write g for the algebra isomorphism $\exp(i_s\circ j)\colon U_1{}^c(Q)\to\mathrm{Sym}(Q)$ of Proposition 10. Since we are using g to transport a bialgebra structure to $\mathrm{Sym}(Q)$, $D_{\mathrm{Sym}(Q)}$ is obtained from the structure map D of $U_1{}^c(Q)$ by conjugation: $D_{\mathrm{Sym}(Q)} = g\circ D\circ g^{-1}$. From the proof of Proposition 10, we have that $g\circ D = i_s\circ j$. Since the image of $i_s\circ j$ is $i_s(Q)$, g induces a bijection from $D(U_1{}^c(Q))$ to $i_s(Q)$. Since $D\circ D = D$ by Proposition 6(iii), we have that $D_{\mathrm{Sym}(Q)}$ is the identity on $i_s(Q)$, and thus that the map $Q \cong Q(\mathrm{Sym}(Q))\to\mathrm{Sym}(Q)$ is the natural inclusion.

Thus, our calculation above shows that, for $x\in Q$, Δx is a sum of Campbell-Hausdorff terms $m_{u,v}[x]_{n-1}$.

We note that our calculation of $\Delta\circ D$ does not rely on the characterization of B as $U_1{}^c(Q(B))$, and thus provides another proof that $Q(B)$ determines B.

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